Exactly Solvable Models in Arbitrary Dimensions

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Abstract

We construct a new class of quasi-exactly solvable many-body Hamiltonians in arbitrary dimensions, whose ground states can have any correlations we choose. Some of the known correlations in one dimension and some recent novel correlations in two and higher dimensions are reproduced as special cases. As specific interesting examples, we also write down some new models in two and higher dimensions with novel correlations.

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Exactly solvable models have always attracted a lot of interest in theoretical physics, because they serve as paradigms for more complicated realistic models. Examples of such many-body quantum Hamiltonians with exact solutions abound in one dimension - $e.g.$, the Calogero-Sutherland model (CSM) [1] and its variants [2, 3] - and they have proved useful in the study of diverse physical phenomena, such as quantum spin chains [4], soliton wave propogation[5] and random matrices[6]. A particular feature of these models is that their wave-functions show strong inter-particle correlations. This feature is expected to be of interest in higher dimensions too, in the study of strongly correlated systems. Hence, the construction of exactly solvable models in higher dimensions is of great interest.

A direct generalisation of the CS correlation built into the wave-function through the Jastrow factor leads to the Laughlin [7] (or Jain[8]) wave-functions in two dimensions, which are relevant to the phenomenon of the fractional quantum Hall effect. Hamiltonians with ultra-short interactions [9] for which Laughlin wave-functions are exact ground state wave-functions have been known for years. We have also been able to construct gauge models [10] for which the Laughlin wave-functions are exact ground states.

Recently, a new form of the pair correlator was constructed in two dimensions[11], which was anti-symmetric under exchange, but also introduced zeroes in the wave-function whenever the relative angle between the position vectors of the two particles was zero or π. The Hamiltonian for which a wave-function constructed from these correlators is a ground state was also constructed and studied. Later, this kind of model was generalised to arbitrary dimensions as well [12]. More recently, Ghosh[13] has constructed new CS type models with only two-body interactions in arbitrary dimensions. Although these models, a priori, may appear to have little or no physical interest at the present moment, there are two reasons for studying such models. One reason is the paucity of exactly solvable models in higher dimensions, which makes such a study a worthwhile exercise. Secondly, and more interestingly, it challenges the ingenuity of the readers to look for realistic systems in physics or in other cross-disciplinary areas, where such novel correlations may be realised.

With this motivation, in this paper, we construct a whole class of exactly solvable models in arbitrary dimensions. The ground state wave-function is chosen to be an arbitrary homogeneous function (multiplied by an exponential factor) of DN variables (D is the dimension and N is the number of

particles), which is symmetric or anti-symmetric under exchange of the N particles, depending on whether we are interested in bosonic or fermionic wave-functions. Then the Hamiltonian is constructed in terms of the partial derivatives of the wave-function. Further, by writing the Hamiltonian in terms of appropriate creation and annihilation operators, a class of excited states and their energies is found.

By taking specific forms of the wave-function, we show that the CSM and its variants in one dimension as well as the more recent models in two and arbitrary dimensions are reproduced as special cases. Interestingly, we find that in two dimensions, we can construct exactly solvable models for which the more natural Jastrow-type correlations are relevant. More specifically, we show that our procedure allows for the construction of a model for which the unprojected Jain wavefunctions are exact ground states. As a final example, we construct a new fermionic Hamiltonian in three dimensions with two and three body interactions which can be exactly solved for the ground state and and a class of excited states. The ground state wave-function of this model has the novel feature that it vanishes not only when two particles coincide, but also when their position vectors differ by multiples of a specified lattice vector (generalising the result of Murthy et al[11] in two dimensions).

Let us consider a system of N particles in D dimensions with position vectors $\mathbf{r}_1 = (x_1^1, x_1^2, \cdots, x_1^D), \mathbf{r}_2 = (x_2^1, x_2^2, \cdots, x_2^D), \ldots, \mathbf{r}_N = (x_N^1, x_N^2, \cdots, x_N^D)$. We now prove a general mathematical theorem which is applicable to this system.

- **Theorem I** : Let f be a function of ND variables (*i.e.*, $f \equiv f(x_i^j)$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, D$) that is homogeneous and of degree n. Then $\psi = f^\eta(x_i^j)e^{-\frac{1}{2}\sum_i^N \sum_j^D x_i^{j2}}$ is[2] an exact eigenstate of the Hamiltonian

$$
H = -\frac{1}{2}\sum_i^N \vec{\nabla}_i^2 \;\; + \;\; \frac{1}{2}\sum_i^N \mathbf{r}_i^2 + \frac{g_1}{2}\sum_i^N \sum_j^D \frac{\partial^2 f/\partial x_i^{j2}}{f}
$$

$$
+ \;\; \frac{g_2}{2}\sum_i^N \sum_j^D (\frac{\partial f/\partial x_i^j}{f})^2 \tag{1}
$$

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[2]Note that $f^\eta = g$ is also a homogeneous function of its variables. Hence, the most general form of the theorem has $\eta = 1$. However, for future convenience in constructing explicit examples, we retain η.

3

when

$$g_1 = \eta \quad \text{and} \quad g_2 = \eta(\eta - 1) \tag{2}$$

and with eigenvalue

$$E = \frac{1}{2}(2\eta n + ND). \tag{3}$$

- **Proof** : Considered as a function of a single variable x_i^j, we see that

$$
\begin{aligned}
\frac{\partial^2 \psi}{\partial x_i^{j2}} &= \eta(\eta - 1)\frac{(\partial f/\partial x_i^j)^2}{f^2}\psi + \eta\frac{\partial^2 f/\partial x_i^{j2}}{f}\psi \\
&- 2\eta x_i^j \frac{\partial f/\partial x_i^j}{f}\psi - \psi + x_i^{j2}\psi.
\end{aligned}
\tag{4}
$$

Summing over all the ND coordinates and using Euler's theorem for homogeneous functions,

$$\sum_i^N \sum_j^D x_i^j \frac{\partial f}{\partial x_i^j} \equiv \sum_{i,j}^N x_i^j \frac{\partial f}{\partial x_i^j} = nf, \tag{5}$$

we get

$$
\begin{aligned}
\sum_{i,j}^N \frac{\partial^2 \psi}{\partial x_i^{j2}} &= \eta(\eta - 1)\sum_{i,j}^N \frac{(\partial f/\partial x_i^j)^2}{f^2}\psi + \eta\sum_{i,j}^N \frac{\partial^2 f/\partial x_i^{j2}}{f}\psi \\
&- 2n\eta\psi - ND\psi + \sum_i^N \mathbf{r}_i^2\psi.
\end{aligned}
\tag{6}
$$

Using this in Eq.(1), we get

$$
\begin{aligned}
H\psi &= \frac{1}{2}[g_2 - \eta(\eta - 1)]\sum_{i,j}^N \frac{(\partial f/\partial x_i^j)^2}{f^2}\psi \\
&+ \frac{1}{2}(g_1 - \eta)\sum_{i,j}^N \frac{\partial^2 f/\partial x_i^{j2}}{f}\psi + \frac{1}{2}(2n\eta + ND)\psi.
\end{aligned}
\tag{7}
$$

Hence, for the choice of g_1 and g_2 given in Eq.(2), ψ is an eigenstate of the Hamiltonian with the eigenvalue given in Eq.(3).

The next step is to prove that ψ is actually a ground state wave-function. For this, we formally introduce creation and annihilation operators as follows -

$$
\begin{aligned}
A_x &= p_x - ix + i\eta\frac{\partial f/\partial x}{f} \\
A_x^\dagger &= p_x + ix - i\eta\frac{\partial f/\partial x}{f}
\end{aligned}
\tag{8}
$$

for each of the coordinates x_i^j. We then find that the Hamiltonian in Eq.(1) can be written as

$$
H = \frac{1}{2}\sum_{i,j}^{N}(A_{x_i^j}^\dagger A_{x_i^j}) + \frac{1}{2}(2\eta n + ND).
\tag{9}
$$

ψ is annihilated by all the $A_{x_i^j}$ and is therefore the ground state of the Hamiltonian with the energy $(2\eta n + ND)/2$. Note however, that $[A_x, A_y^\dagger]$ for different coordinates x and y do not commute for non-zero η and hence, the Hamiltonian is not fully soluble for the excited states.

We can also look for excited states of the Hamiltonian by making the following ansatz[11] for the excited state wave-function -

$$
\Phi(x_i^j) = \psi_0(x_i^j)\phi(x_i^j)
\tag{10}
$$

- *i.e.*, we assume that the excited state wave-function factorises in terms of the ground state wave-function ψ_0 and a completely symmetric wave-function ϕ. This form can be justified by looking at the asymptotic properties of the Hamiltonian.

Using the ansatz in Eq.(10) in the original Hamiltonian, we find that $\phi(x_i^j)$ satisfies

$$
-\frac{1}{2}\sum_{i,j}^{N}\frac{\partial^2\phi}{\partial x_i^{j2}} + \sum_{i,j}^{N}x_i^j\frac{\partial\phi}{\partial x_i^j} - \eta\sum_{i,j}^{N}\frac{\partial f/\partial x_i^j}{f}\frac{\partial\phi}{\partial x_i^j} = (E_m - E_0)\phi
\tag{11}
$$

Here, we have assumed that

$$
H\Phi = E_m\Phi
\tag{12}
$$

following [11]. We now specialise to the case of radial excitations. - *i.e.*, we look for solutions where ϕ is only a function of $t = \sum_{i,j}^{N}x_i^{j2}$. We then find

5

that the eigenvalue equation for ϕ reduces to the confluent hypergeometric equation given by

$$t\frac{d^2\phi}{dt^2} + (E_0 - t)\frac{d\phi}{dt} + \frac{1}{2}(E_m - E_0)\phi = 0. \tag{13}$$

The solutions are confluent hypergeometric functions which are normalisable provided

$$\frac{1}{2}(E_m - E_0) = m \tag{14}$$

where m is a positive integer. This gives the energy spectrum of radially excited states as

$$E_m = E_0 + 2m. \tag{15}$$

Note that we have obtained the results for the ground state wave-function and the excited states in analogy with Ref.[11]; however, our results are much more general and are valid for *any* homogeneous function f. In fact, for the function $\psi = f^\eta e^{-\frac{1}{2}\sum_i^N \sum_j^D x_i^{j2}}$ to qualify as an N-particle wave-function, f^η must be symmetric/antisymmetric under exchange for bosons/fermions. This is a requirement that we need to impose on f. Furthermore, there exists no restriction as to dimensionality, number of particles, types of correlation, or whether there exists two-body, three-body or N-body interactions in the potential. But for useful results, we need to put some constraints on f, such that either the type of correlation present in the wave-function or the type of interaction present in the potential is specified. In either case, we can construct a whole host of models with exact ground states. But before we go on to the construction of new models, let us first check that by choosing appropriate forms of f, several earlier known models are reproduced as special cases within this formalism.

- (I) One dimensional models (D=1)

    - a)Let us choose f to be of the form

    $$f = \prod_{i<j}^N (x_i - x_j)^\lambda |x_i - x_j|^\alpha \quad \text{and} \quad \psi = f e^{-\sum_i^N \frac{x_i^2}{2}}. \tag{16}$$

    This is the so-called Jastrow form and is a highly correlated wave-function. It picks up a phase $(-1)^\lambda$ under exchange of two particles

6

and can be chosen to be fermionic or bosonic by choosing $\lambda = 1$ or $\lambda = 0$ respectively. By differentiating f, we can check that

$$\sum_i^N \frac{\partial^2 f/\partial x_i^2}{f} = 2\sum_{i<j}^N \frac{(\alpha+\lambda)(\alpha+\lambda-1)}{(x_i-x_j)^2} \tag{17}$$

and the three-body term vanishes, since (without loss of generality), we may choose $x_1 < x_2 < \cdots < x_N$ and use the identity

$$\sum_{i\neq j\neq k}^N \frac{1}{(x_i-x_j)(x_i-x_k)} = 0. \tag{18}$$

Thus for this value of f, the Hamiltonian in Eq.(1) yields the Calogero-Sutherland model.

- b)A simple generalisation is to choose f to be of the form

$$f = \prod_{i<j}^N (x_i^k - x_j^k)^\lambda |x_i^k - x_j^k|^\alpha \tag{19}$$

and construct the Hamiltonian. ($k = 1$ is the CSM.) For $k = 2$, the three body-term vanishes[13]. But for other values of k, we have both two-body and three-body interactions.

In principle, we can construct many further models with multi-particle correlations, since f is arbitrary. For fermionic wave-functions, f has to be antisymmetric under exchange, which can always be achieved by choosing f to be of the form of a determinant multiplied by symmetric factors.

- (II) Two dimensional models (D=2)

  Here again, there are several possibilities.

  - (a) As an obvious generalisation of the one dimensional Jastrow form, we may take

$$f = \prod_{i<j}^N (|\mathbf{x}_i| - |\mathbf{x}_j|)^\lambda ||\mathbf{x}_i| - |\mathbf{x}_j||^\alpha.$$
$$\text{and} \quad \psi = f e^{-\frac{1}{2}\sum_{i,j}^N x_i^{j2}} \tag{20}$$

7

where $\mathbf{x}_i = (x_i^1, x_i^2)$ and f is antisymmetric under exchange when λ is an odd integer. The Hamiltonian is easily constructed just as in the one dimensional case. The three-body term vanishes by the same logic that it vanishes for the one-dimensional case - *i.e.*, since $|\mathbf{x}_i|$ is a number, the phase space can be split into identical copies and within each copy, it is easy to show that the three-body interaction term vanishes by the identity in Eq.(18).

– (b) Recently, a new type of pair correlator was found in two dimensions with which a Jastrow type correlator could be constructed. This was of the form $x_i y_j - x_j y_i$. Using this, one can construct the many-body wave-function

$$f = \prod_{i<j}^N (x_i y_j - x_j y_i)^\lambda |x_i y_j - x_j y_i|^\alpha \qquad (21)$$

and the corresponding Hamiltonian, which was earlier written down and studied in Ref.[11].

– (c) We may also choose f to be of the form

$$f = \prod_{i<j}^N (|\mathbf{x}_i|^k - |\mathbf{x}_j|^k)^\lambda . ||\mathbf{x}_i|^k - |\mathbf{x}_j|^k|^\alpha \qquad (22)$$

In this case, using our general procedure, we can write down the explicit models. Once again, since $|\mathbf{x}_i|$ is a number, it is easy to check that the three-body term vanishes for $k = 2$ (besides $k = 1$, of course) and hence for this case, a CS type model can be constructed. This model was recently constructed by Ghosh [13], as an example of a CS type model in higher dimensions.

• (III) Three-dimensional models (D=3)

Here, again, some of the lower dimensional models can be obviously generalised.

– (a) A direct generalisation of type (c) models in two dimensions yields the wave-function

$$\psi = \prod_{i<j}^N (|\mathbf{x}_i|^k - |\mathbf{x}_j|^k)^\lambda . ||\mathbf{x}_i|^k - |\mathbf{x}_j|^k|^\alpha e^{-\frac{1}{2}\sum_{i,j}^N x_i^{j2}} \qquad (23)$$

8

for which Hamiltonians can be constructed. As before, for $k = 1$ and $k = 2$, three-body terms vanish.

– (b) A generalisation of the type (b) models has also been earlier considered[12]. The simplest way is to consider that the two vector determinant is now replaced by the three vector determinant, *i.e.*,

$$P_{ij} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} \longrightarrow \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix} = P_{ijk} \tag{24}$$

($\mathbf{x}_i = (x_i, y_i)$ in two dimensions and $\mathbf{x}_i = (x_i, y_i, z_i)$ in three dimensions). f can now be constructed as $f = (P_{ijk})^\lambda |P_{ijk}|^\alpha$.

Some models have been considered in even higher dimensions, but we shall not consider them explicitly, since our point is merely to illustrate that all models with polynomial correlations can be obtained from our general procedure.

Finally, we construct new models in two and three dimensions using our general procedure.

- (i) D=2

  Choose ψ to be the unprojected Jain wave-function given by

$$\psi = \prod_{i<j}^{N} (z_i - z_j)^{2m} \chi_n e^{-\frac{1}{2} \sum_i^N z_i z_i^*} \tag{25}$$

  where we have used complex notation and χ_n is the Slater determinant of n filled Landau levels - *i.e.*, χ_n is a determinant involving at most $n-1$ powers of z_i^* in each of its terms. For this Jastrow-Slater form, the wave-function does not factorise neatly in the usual Jastrow form. Still by using $f = \prod_{i<j}^{N} (z_i - z_j)^{2m} \chi_n$ and taking the appropriate derivatives as in Eq.1, we can construct the Hamiltonian, which, however, has many body interactions and is not simple.

9

- (ii) D=3

  Choose f to be of the following form

  $$
  f = \prod_{i<j}^{N} \begin{vmatrix} 1 & x_i & x_j \\ 1 & y_i & y_j \\ 1 & z_i & z_j \end{vmatrix}^{\lambda} \times \mathrm{abs} \begin{vmatrix} 1 & x_i & x_j \\ 1 & y_i & y_j \\ 1 & z_i & z_j \end{vmatrix}^{\alpha}
  $$

  $$
  \equiv \prod_{i<j}^{N} X_{ij}^{\lambda} \times |X_{ij}|^{\alpha}
  $$

  $$
  \text{and} \quad \psi = f e^{-\frac{1}{2} \sum_i^N (x_i^2 + y_i^2 + z_i^2)} \tag{26}
  $$

  A fermionic model can always be constructed by choosing λ to be an odd integer, since the determinant enforces antisymmetry under exchange. However, note that the wave-function vanishes not only when the relative angle between the two position vectors is zero or π (as in Ref.[11], but whenever the two position vectors satisfy $\mathbf{r}_i = m\mathbf{r}_j + k(1, 1, 1)$ (m, k are integers) due to the property of the determinant. This wave-function obviously has only two-body correlations and can be thought of as yet another generalisation of case (b) in two dimensions. The Hamiltonian can be constructed from this wave-function by our general procedure and is given by

  $$
  \begin{aligned}
  H = {} & -\frac{1}{2} \sum_i^N \vec{\nabla}_i^2 + \frac{1}{2} \sum_i^N \mathbf{r}_i^2 \\
  & - \frac{(\alpha + \lambda)}{2} \sum_{i, j \neq i} \frac{(y_j - z_j)^2 + (z_j - x_j)^2 + (x_j - y_j)^2}{X_{ij}^2} \\
  & + \frac{(\alpha + \lambda)^2}{2} \sum_i^N \left[\sum_{j \neq i}^N \frac{(y_j - z_j)}{X_{ij}} \right)^2 + \left(\sum_{j \neq i}^N \frac{(z_j - x_j)}{X_{ij}} \right)^2 \\
  & + \left(\sum_{j \neq i}^N \frac{(x_j - y_j)}{X_{ij}} \right)^2 \ \right]. \tag{27}
  \end{aligned}
  $$

  This Hamiltonian clearly has both two-body and three body interactions. Its energy is given by $E = ((\alpha + \lambda)N(N - 1) + 3N/2)$.

- (iii) Generalisations

    Simple generalisations of this model with two-body correlations where

$$\begin{vmatrix} 1 & x_i & x_k \\ 1 & y_i & y_k \\ 1 & z_i & z_k \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & x_i^n & x_k^n \\ 1 & y_i^n & y_k^n \\ 1 & z_i^n & z_k^n \end{vmatrix} \tag{28}$$

    or models with three particle correlations where

$$\begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{vmatrix} \longrightarrow \begin{vmatrix} x_i^n & x_j^n & x_k^n \\ y_i^n & y_j^n & y_k^n \\ z_i^n & z_j^n & z_k^n \end{vmatrix} \tag{29}$$

    are also possible.

However, our aim is not to write out an exhaustive set of models. It is more to demonstrate that our general procedure allows the construction of explicitly soluble Hamiltonians. For any specified correlation of the ground state wave-function, we have a systematic procedure to construct the Hamiltonian. And for all these Hamiltonians, we can construct a class of excited states based on radial excitations, using the ansatz in Eq.(10).

To conclude, in this letter, we have constructed a class of non-trivial Hamiltonians, which can be solved exactly for the ground state and a class of excited states. The ground state can have any correlation we choose. In particular, we find that any polynomial correlation including the Jastrow and several recent novel correlations can be reproduced as special cases. We also construct new exactly solvable fermionic Hamiltonians in two and three dimensions as an illustration of our general procedure.

Acknowledgments

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